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The Zeeman effect for the relativistic bound state

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Abstract. In the context of a relativistic quantum mechanics with invariant evolution parameter, solutions for the relativistic bound-state problem have been found, which yield a spectrum for the total mass coinciding with the non-relativistic Schrödinger energy spectrum. These spectra were obtained by choosing an arbitrary spacelike unit vector n_μ and restricting the support of the eigenfunctions in spacetime to the subspace of the Minkowski measure space, for which $(x_\perp)^2 = [x - (x \cdot n)n]^2 \geqslant 0$. In this paper, we examine the normal Zeeman effect (in lowest order) for these bound states, which requires n_μ to be a dynamical quantity. We recover the usual Zeeman splitting in a manifestly covariant form.

1. Introduction

A relativistically covariant quantum mechanical formulation of the two-body bound-state problem has posed serious problems for many years. The Bakamjian–Thomas [1] formalism, using the square-root relation between energy and momentum with the particle on mass-shell, for example, poses the difficulty of separating centre-of-mass motion for local potentials, in addition to analyticity problems and questions of how to define the interaction terms (generally, one uses a power series expansion of the square-root expressions [2]).

A formulation of this problem in terms of Stueckelberg's method of using off-shell kinematics [3], with an invariant evolution parameter, generalized to the case of two or more particles by Horwitz and Piron [4], was shown to yield an understanding of the Schrödinger spectrum for (spinless) hydrogen (and other central potential problems), with relativistic corrections, by Arshansky and Horwitz [5]. The wavefunctions provided by this method are exact solutions of a Poincaré invariant Schrödinger-type equation, and form an (induced) representation of the Lorentz group [6]. Although extending the symmetry from the $O(3)$ of the non-relativistic problem to the $O(3,1)$ of the relativistic case introduces new quantum numbers, the spectrum is degenerate with respect to these new degrees of freedom if they are not coupled through new interactions, as in the models treated above. Moreover, dipole radiation, emitted in transitions among these bound states, obeys selection rules which are formally identical to those of the non-relativistic problem but with covariant interpretation [7].

The appearance of a selection rule for the 'magnetic' quantum number—with respect to which the spectrum is degenerate—leads us to consider lifting this degeneracy through the normal Zeeman effect. In the non-relativistic treatment of the Zeeman effect, the degeneracy of the energy levels, associated with the rotation invariance of the Hamiltonian, is lifted by placing the state in an external magnetic field which provides a preferred direction in space and breaks the rotation invariance. In the relativistic treatment presented here, the degeneracy of the spectrum is associated with the Lorentz invariance of the evolution operator, and may be lifted by placing the state in an external magnetic field which provides a preferred direction in spacetime. In this paper we shall demonstrate how, to lowest order, we may couple the magnetic field to the orbital angular momentum of the bound state and obtain a covariant formulation of the usual Zeeman splitting. Since the angular momentum operators for the bound states are in the rotation subgroup of the induced Lorentz group, the derivation of the normal Zeeman effect provides an insight into the geometry of the induced representation.

It has been shown [5] that the replacement

$$r = \sqrt{(\boldsymbol{r}_1 - \boldsymbol{r}_2)^2} \quad \longrightarrow \quad \rho = \sqrt{(\boldsymbol{r}_1 - \boldsymbol{r}_2)^2 - (t_1 - t_2)^2} \tag{1.1}$$

in the argument of the usual central force potentials of non-relativistic mechanics leads to a relativistic problem, whose Galilean limit is the original non-relativistic central force problem (the correspondence is established by the fact that $t_1 \to t_2$ in this limit). In the context of the relativistic quantum mechanics with invariant evolution parameter [3, 4, 8] referred to above, the resulting mass spectrum coincides with the non-relativistic Schrödinger energy spectrum. It then follows, as we show below, that for small excitations, the corresponding energy spectrum is that of the non-relativistic Schrödinger theory with relativistic corrections. These spectra are obtained when one chooses a spacelike unit vector n_μ ($g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1) \;\Rightarrow\; n^2 = +1$) and restricts the support of the eigenfunctions in spacetime to the subspace of the Minkowski measure space corresponding to the condition

$$(x_\perp)^2 = [x - (x \cdot n)n]^2 \geqslant 0 \tag{1.2}$$

where we denote by $x \equiv x^\mu$ the *relative* coordinates $x_1^\mu - x_2^\mu$, for the two-body system, and $x^2 = x^\mu x_\mu$. The restricted space, called the RMS (restricted Minkowski space), is transitive and invariant under the $O(2, 1)$ subgroup of $O(3, 1)$ leaving n_μ invariant and translations along n_μ.

The two-body (Poincaré invariant) Hamiltonian in this theory,

$$K = \frac{p_{1\mu} p_1^\mu}{2M_1} + \frac{p_{2\mu} p_2^\mu}{2M_2} + V(\rho) \tag{1.3}$$

is quadratic in the 4-momenta, and one may separate variables of the centre-of-mass motion and relative motion in the same way as in the non-relativistic theory,

$$K = \frac{P^\mu P_\mu}{2M} + \frac{p^\mu p_\mu}{2m} + V(\rho) \tag{1.4}$$

where

$$\begin{aligned} P^\mu &= p_1^\mu + p_2^\mu & \qquad M &= M_1 + M_2 \\ p^\mu &= (M_2 p_1^\mu - M_1 p_2^\mu)/M & \qquad m &= M_1 M_2 / M\,. \end{aligned} \tag{1.5}$$

In [5], n_μ was chosen to be the z-axis, and the relative Hamiltonian

$$K_{rel} = \frac{p^\mu p_\mu}{2m} + V(\rho) \tag{1.6}$$

was expressed in terms of coordinates with the parametrization

$$\begin{aligned} y^0 &= \rho \sinh\beta \sin\theta & \qquad y^1 &= \rho \cosh\beta \sin\theta \cos\phi \\ y^2 &= \rho \cosh\beta \sin\theta \sin\phi & \qquad y^3 &= \rho \cos\theta \end{aligned} \tag{1.7}$$

for which

$$(y^1)^2 + (y^2)^2 - (y^0)^2 \geqslant 0\,. \tag{1.8}$$

It was shown in [5, 6] that the eigenfunctions of K_{rel} form irreducible representations of $SU(1,1)$—in the double covering of $O(2,1)$—parametrized by the spacelike vector n_μ stabilized by this particular $O(2,1)$. We remark that the spectrum of K_{rel} emerges in measurements of the total two-body energy of the system through the relations (1.4) and (1.6). In the centre-of-momentum frame, only the total energy contributes to the first term of (1.4), which we can then write as (we write the velocity of light c explicitly here)

$$E^2 = 2mc^2(K_{rel} - K)\,. \tag{1.9}$$

Using the asymptotic value $K \approx -\frac{mc^2}{2}$, obtained from (1.3) if $V(\rho)$ is negligible and the particles are asymptotically 'on shell', i.e. $p_i^\mu p_{i\mu} \approx -m_i^2 c^2$, the energy spectrum, for values K'_{rel} of K_{rel} small compared to the total mass, is given by

$$E \cong mc^2 + K'_{rel} + \mathrm{o}\left(\frac{K'^2_{rel}}{mc^2}\right). \tag{1.10}$$

The terms in $\mathrm{o}\left(\frac{K'^2_{rel}}{mc^2}\right)$ are relativistic corrections.

In [6], an induced representation of $SL(2,C)$ was constructed, by applying the Lorentz group to the RMS coordinates x^μ and the frame orientation n_μ, and studying the action on these wavefunctions. One first observes that wavefunctions with support on

$$x \in \mathrm{RMS}(n_\mu) = \left\{x \mid [x - (x\cdot n)n]^2 \geqslant 0\right\} \tag{1.11}$$

may be written as functions of n_μ and the coordinates of a standard frame $y \in \mathrm{RMS}(\mathring{n}_\mu)$ since, given the Lorentz transformation $\mathcal{L}$ such that $\mathring{n} = \mathcal{L}(n)\,n$, it follows that

$$x \in \mathrm{RMS}(n_\mu) \qquad \text{and} \qquad y = \mathcal{L}(n)\,x \Longrightarrow y \in \mathrm{RMS}(\mathring{n}_\mu)\,. \tag{1.12}$$

By choosing $\mathring{n} = (0,0,0,1)$ as in [5], the parametrization (1.7) may be used for y^μ. Now, under Lorentz transformations labelled by Λ, the wavefunctions were shown to transform as

$$\psi_n(y) \to \psi_n^\Lambda(y) = \psi_{\Lambda^{-1}n}(D^{-1}(\Lambda,n)\,y) \tag{1.13}$$

where Λ acts directly on n_μ. The representations are moved on an orbit generated by this spacelike vector, and the Lorentz transformations act on y^μ through the $O(2,1)$ little group, represented by $D^{-1}(\Lambda, n)$, with the property

$$D^{-1}(\Lambda,n)\,\mathring{n} = \mathcal{L}(\Lambda n)\,\Lambda\,\mathcal{L}^T(n)\,\mathring{n} \equiv \mathring{n}\,. \tag{1.14}$$

The matrix $\mathcal{L}^T(n)$ was chosen in [6] to be a boost in the three-direction, a rotation about the two-axis, followed by a rotation about the 1-axis. Thus,

$$\mathcal{L}^T(n) = \mathrm{e}^{\gamma\mathcal{M}^{23}}\mathrm{e}^{\omega\mathcal{M}^{31}}\mathrm{e}^{\alpha\mathcal{M}^{03}} \tag{1.15}$$

where

$$(\mathcal{M}^{\sigma\lambda})^{\mu\nu} = g^{\sigma\mu}g^{\lambda\nu} - g^{\sigma\nu}g^{\lambda\mu} \tag{1.16}$$

and so

$$\mathcal{L}^T(n) = \begin{pmatrix} \cosh\alpha & 0 & 0 & \sinh\alpha \\ -\sin\omega\,\sinh\alpha & \cos\omega & 0 & -\sin\omega\,\cosh\alpha \\ \sin\gamma\,\cos\omega\,\sinh\alpha & \sin\gamma\,\sin\omega & \cos\gamma & \sin\gamma\,\cos\omega\,\cosh\alpha \\ \cos\gamma\,\cos\omega\,\sinh\alpha & \cos\gamma\,\sin\omega & -\sin\gamma & \cos\gamma\,\cos\omega\,\cosh\alpha \end{pmatrix} \tag{1.17}$$

which provides the parametrization of n_μ as

$$n_\mu = \begin{pmatrix} \sinh\alpha \\ -\sin\omega\,\cosh\alpha \\ \sin\gamma\,\cos\omega\,\cosh\alpha \\ \cos\gamma\,\cos\omega\,\cosh\alpha \end{pmatrix}. \tag{1.18}$$

By examining the generators $h_{\alpha\beta}(n)$ of (1.13), which form a representation of the $O(3,1)$ Lie algebra (through their action on y and n), the Casimir operators

$$\hat{c}_1 = \tfrac{1}{2} h_{\alpha\beta}(n) h^{\alpha\beta}(n) \qquad \hat{c}_2 = \tfrac{1}{2} \epsilon^{\alpha\beta\gamma\delta} h_{\alpha\beta}(n) h_{\gamma\delta}(n) \tag{1.19}$$

as well as the operators of the $SU(2)$ subgroup

$$L^2(n) = \tfrac{1}{2} h_{ij}(n) h^{ij}(n) \qquad L_1(n) = h^{23}(n) = -\mathrm{i}\frac{\partial}{\partial\gamma} \tag{1.20}$$

can be constructed as a commuting set. Moreover, the operator

$$\Lambda = \tfrac{1}{2} M^{\mu\nu} M_{\mu\nu} \rightarrow \ell(\ell+1) - \tfrac{3}{4} \tag{1.21}$$

where $M^{\mu\nu} = y^\mu p^\nu - y^\nu p^\mu$, and the $O(2,1)$ Casimir $N^2 = (M^{01})^2 + (M^{02})^2 + (M^{12})^2$ commute with this set. Wavefunctions were then constructed which are eigenfunctions of the set

$$\{\Lambda, N^2, \hat{c}_1, \hat{c}_2, L^2(n), L_1(n)\} \tag{1.22}$$

with eigenvalues $Q = \{\ell(\ell+1) - \frac{3}{4}, n^2 - \frac{1}{4}, c_1, c_2, L(L+1), q\}$. The requirement that these wavefunctions lie in a unitary irreducible representation of $SL(2,C)$ (they are in the principal series), imposes the condition $c_1 = \hat{n}^2 - 1 - c_2^2/\hat{n}^2$, where $\hat{n} = n + \frac{1}{2}$.

The remaining 'radial' function, after the transformation $\hat{R}(\rho) = R(\rho)/\sqrt{\rho}$ of the radial part of $\psi_n(y)$, must then satisfy an equation which is precisely of the form of the non-relativistic Schrödinger radial equation in three dimensions (and has the same normalization). The states $\psi_n(y)$ are then eigenstates of the Lorentz invariant K_{rel}, whose support is on RMS(n), with the quantum numbers (1.22), and a principal quantum number n_a. In particular, the solutions for the problem corresponding to the Coulomb potential [5] yield bound states with a mass spectrum which coincides with the non-relativistic Schrödinger energy spectrum. The observed *energies* for such systems are determined by the values of $P^\mu P_\mu$, i.e. $-E^2$ in the centre-of-momentum frame; from (1.4) one obtains, as noted in (1.10), in an expansion in orders of $1/c^2$, the non-relativistic spectrum with relativistic corrections.

The selection rules for dipole radiation from these states have been calculated [7] and have been shown to be identical with those of the usual non-relativistic theory, expressed in a manifestly covariant form,

$$\{\Delta\ell = \pm 1;\ \Delta q = 0, \pm 1\}. \tag{1.23}$$

In addition to the transverse and longitudinal polarizations of the non-relativistic theory, there is a 'scalar' transition, induced by the relative time coordinate. The 'scalar' polarization and the longitudinal polarization induce the same $\Delta q = 0$ transition for the relativistic case, which has a natural interpretation in terms of the Gupta–Bleuler quantization of the photon. This relationship shows that the wavefunctions act correctly as representations of the the Lorentz group. Moreover, it was shown in [7] that the change in q, the eigenvalue of $L_1(n)$, corresponds to a change in the orientation of n_μ with respect to the polarization of the emitted or absorbed photon. That the magnetic quantum number q depends on the frame orientation should not be surprising, because the operator $L_1(n)$ belongs to the $SU(2)$ subgroup of $SL(2,C)$, and acts on n_μ, but not on the RMS coordinates (it was shown in [7] that for Λ a rotation about the 1-axis, $D^{-1}(\Lambda, n) \equiv 1$).

In this paper, we provide a derivation of the Zeeman effect for the bound states, which requires allowing n_μ to become a dynamical quantity. We begin with a discussion of the classical $O(3,1)$ in the induced representation and obtain the group generators, which coincide with those of [6], when the momenta are understood as derivatives in the Poisson bracket sense. We construct a classical Lagrangian, in which n_μ plays an explicit dynamical role, and show that the generators are conserved. We then construct the Hamiltonian, which may be unambiguously quantized and made locally gauge invariant. Finally, it is shown that an external gauge field representing a constant magnetic field induces a mass level splitting corresponding to the usual non-relativistic expression.

2. The configuration space

We shall be interested, in this section, in the classical relativistic mechanics of events of spacelike separation. We characterize the separation vectors by the coordinates (n, y), where n is the spacelike unit vector parametrized in (1.18); $y \in \mathrm{RMS}(\mathring{n})$ is parametrized in (1.7) (note that $\mathcal{L}^T(n)y \in \mathrm{RMS}(n)$) and satisfies (1.2).

Under a Lorentz transformation Λ, we know that

$$n \to n' = \Lambda n \qquad x \to x' = \Lambda x . \tag{2.1}$$

It follows from (1.12) and (2.1) that

$$x' = \Lambda x = \Lambda \mathcal{L}(n)^T y = \mathcal{L}(\Lambda n)^T \mathcal{L}(\Lambda n) \Lambda \mathcal{L}(n)^T y = \mathcal{L}(n')^T y' . \tag{2.2}$$

Thus y transforms as

$$y \to y' = D^{-1}(\Lambda, n) y \tag{2.3}$$

where (as in (1.14)) $D^{-1}(\Lambda, n) = \mathcal{L}(\Lambda n) \Lambda \mathcal{L}(n)^T$ belongs to the $O(2, 1)$ which leaves $\mathring{n}$ invariant, i.e.

$$D^{-1}(\Lambda, n) \mathring{n} = \mathcal{L}(\Lambda n) \Lambda \mathcal{L}(n)^T \mathring{n} = \mathring{n} \tag{2.4}$$

and hence the relation (1.8) is preserved. The coordinates thus transform as

$$\Lambda : (n, y) \quad \to \quad (n, y)' = (\Lambda n, D^{-1}(\Lambda, n) y) . \tag{2.5}$$

We wish now to construct a model for the Zeeman effect in this covariant framework. To do this, we recall that in the computation of the selection rules for radiative processes, as we remarked above, the restriction $\Delta q = 0, \pm 1$ refers to a reaction of the radiation on the orientation of the coset label n^μ of the induced representation. In the dipole approximation, the transition operator is x^μ, and in [7], we demonstrated that the conservation of the eigenvalues L and n in the matrix elements of x^μ implies the vanishing of the matrix element $\langle \ell' n' | \sin\theta | \ell n \rangle$, leaving only the terms containing $\langle \ell' n | \cos\theta | \ell n \rangle$ in the calculations. Since this term arises only from the $y^3 = \rho \cos\theta$ component of y^μ, the terms of x^μ which contribute to these matrix elements are of the form $\mathcal{L}(n)^{3\mu} y_3$. The 3-column of $\mathcal{L}^T$ is precisely n_μ, so the calculation factors as

$$\begin{aligned} \langle n_{a'} \ell' n' L' q' c_2' | x^\mu | n_a \ell n L q c_2 \rangle &= \langle n_{a'} \ell' n' L' q' c_2' | \rho \cos\theta \, n^\mu | n_a \ell n L q c_2 \rangle \\ &= \langle n_{a'} \ell' | \rho | n_a \ell \rangle \langle \ell' n | \cos\theta | \ell n \rangle \langle n' L' q' c_2' | n^\mu | n L q c_2 \rangle \\ &= \langle n_{a'} \ell' | \rho | n_a \ell \rangle \langle \ell' n | \cos\theta | \ell n \rangle \, \langle n L q' c_2 | n^\mu | n L q c_2 \rangle \\ &\quad \times \delta_{nn'} \, \delta_{LL'} \, \delta(c_2 - c_2') . \end{aligned} \tag{2.6}$$

Since $|n_a \ell\rangle$ refers to the radial functions and the functions $|\ell n\rangle$ are the usual spherical harmonics, (2.6) shows directly that it is the orientation of n_μ which determines the transition in q.

We deduce from this result that the vector n^μ must be effectively coupled to the radiation field, and we shall build our model for coupling to the electromagnetic field by adding to the Lagrangian a kinetic term for the evolution of n^μ which, with minimal gauge invariance, provides the Zeeman coupling.

The *velocity* $\dot{n} = \mathrm{d}n/\mathrm{d}\tau$ transforms just as n does, since τ is invariant:

$$n' = \Lambda\, n \quad \Longrightarrow \quad \dot{n}' = \Lambda\, \dot{n} \tag{2.7}$$

but since $\mathcal{L}(n)$ is now τ-dependent, the transformation of $\dot{y}$ is more complicated. We may write

$$y = \mathcal{L}(n(\tau))\, x \quad \Longrightarrow \quad \dot{y} = \mathcal{L}(n)\dot{x} + \dot{\mathcal{L}}(n)x \tag{2.8}$$

$$x = \mathcal{L}(n(\tau))^T\, y \quad \Longrightarrow \quad \dot{x} = \mathcal{L}(n)^T\dot{y} + \dot{\mathcal{L}}(n)^T y \tag{2.9}$$

and we see that since $\mathrm{d}/\mathrm{d}\tau$ and the Lorentz transformation commute, (2.8) is, in fact, form invariant:

$$\begin{aligned}
(\dot{y})' &= \mathcal{L}(n')\dot{x}' + \dot{\mathcal{L}}(n')x' \\
&= \mathcal{L}(\Lambda n)[\Lambda\dot{x}] + \dot{\mathcal{L}}(\Lambda n)[\Lambda x] \\
&= \mathcal{L}(\Lambda\, n)\Lambda[\mathcal{L}(n)^T\dot{y} + \dot{\mathcal{L}}(n)^T y] + \dot{\mathcal{L}}(\Lambda n)[\Lambda\mathcal{L}(n))^T\, y] \\
&= [\mathcal{L}(\Lambda n)\Lambda\mathcal{L}(n)^T]\dot{y} + [\mathcal{L}(\Lambda n)\Lambda\dot{\mathcal{L}}(n)^T + \dot{\mathcal{L}}(\Lambda\, n)\Lambda\mathcal{L}(n))^T]\, y \\
&= D^{-1}(\Lambda, n)\dot{y} + \dot{D}^{-1}(\Lambda, n)\, y \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}[D^{-1}(\Lambda, n)\, y]\,.
\end{aligned} \tag{2.10}$$

The *phase space* (which must include n and $\dot{n}$) transforms as

$$\Lambda : \{(n, y); (\dot{n}, \dot{y})\} \longrightarrow \{(\Lambda n, D^{-1}(\Lambda, n)y); (\Lambda\dot{n}, D^{-1}(\Lambda, n)\dot{y} + \dot{D}^{-1}(\Lambda, n)y)\}\,. \tag{2.11}$$

We now examine the generators of the Lorentz transformation represented in (2.5). We take

$$\Lambda = 1 + \lambda + \mathrm{o}(\lambda^2) \tag{2.12}$$

and write λ as

$$\lambda = \tfrac{1}{2}\omega_{\alpha\beta}\mathcal{M}^{\alpha\beta} \tag{2.13}$$

where $\omega_{\alpha\beta}$, $\alpha, \beta = 0, \ldots, 3$ is (infinitesimal) antisymmetric. The matrix generators

$$\mathcal{M}^{\alpha\beta} = \left.\frac{\partial\lambda}{\partial\omega_{\alpha\beta}}\right|_{\omega=0} \tag{2.14}$$

are those given in (1.16). According to (2.12) and (2.13), (2.5) becomes

$$\Lambda : (n, y) \quad \to \quad (n, y)' = (n + \lambda n, \mathcal{L}(n + \lambda n)(1 + \lambda)\mathcal{L}(n)^T y) + \mathrm{o}(\omega^2)\,. \tag{2.15}$$

Defining the generators of $\xi = (n, y) \to \xi' = (n', y')$ as

$$X_{\alpha\beta} = \sum_{i=1}^{8} \left.\frac{\partial\xi^i}{\partial\omega^{\alpha\beta}}\right|_{\omega=0} \frac{\partial}{\partial\xi^i} \tag{2.16}$$

where for $i = 1, \cdots, 4$, $\xi^i = n^\mu$, $\mu = 0, \ldots, 3$, and for $i = 5, \ldots, 8$, $\xi^i = y^\mu$, $\mu = 0, \ldots, 3$. Thus, for $i = 1, \ldots, 4$,

$$\begin{aligned}
\left.\frac{\partial\xi^i}{\partial\omega^{\alpha\beta}}\right|_{\omega=0} &= \frac{\partial}{\partial\omega^{\alpha\beta}}(n^i + (\lambda n)^i)\big|_{\omega=0} \\
&= \frac{\partial}{\partial\omega^{\alpha\beta}}\left(n^i + \left(\tfrac{1}{2}\omega^{\sigma\rho}\mathcal{M}_{\sigma\rho}n\right)^i\right)\Big|_{\omega=0} \\
&= \tfrac{1}{2}(\delta^\sigma_\alpha\delta^\rho_\beta - \delta^\sigma_\beta\delta^\rho_\alpha)(\mathcal{M}_{\sigma\rho}n)^i \\
&= (\mathcal{M}_{\alpha\beta})^i{}_j n^j
\end{aligned} \tag{2.17}$$

so that

$$\sum_{i=1}^{4} \frac{\partial \xi^i}{\partial \omega^{\alpha\beta}}\bigg|_{\omega=0} \frac{\partial}{\partial \xi^i} = (\mathcal{M}_{\alpha\beta})^{\mu}{}_{\nu} n^{\nu} \frac{\partial}{\partial n^{\mu}}$$
$$= (g^{\mu}_{\alpha} g_{\beta\nu} - g^{\mu}_{\beta} g_{\alpha\nu}) n^{\nu} \frac{\partial}{\partial n^{\mu}}$$
$$= n_{\beta} \frac{\partial}{\partial n^{\alpha}} - n_{\alpha} \frac{\partial}{\partial n^{\beta}} \tag{2.18}$$

which was called $d(\lambda_{\alpha\beta})$ in [6].

Now for $i = 5, \ldots, 8$,

$$\begin{aligned}
\frac{\partial \xi^i}{\partial \omega^{\alpha\beta}}\bigg|_{\omega=0} &= \frac{\partial}{\partial \omega^{\alpha\beta}} \left[\mathcal{L}(n+\lambda n)(1+\lambda)\mathcal{L}(n)^T y\right]^i \bigg|_{\omega=0} \\
&= \left[\frac{\partial}{\partial \omega^{\alpha\beta}} \mathcal{L}(n+\lambda n)\big|_{\omega=0} \mathcal{L}(n)^T y\right]^i + \left[\mathcal{L}(n) \frac{\partial}{\partial \omega^{\alpha\beta}} \lambda\big|_{\omega=0} \mathcal{L}(n)^T y\right]^i \\
&= \left[\frac{\partial}{\partial n^{\mu}} \mathcal{L}(n) \frac{\partial}{\partial \omega^{\alpha\beta}} (\lambda n)^{\mu}\big|_{\omega=0} \mathcal{L}(n)^T y\right]^i + \left[\mathcal{L}(n) \mathcal{M}_{\alpha\beta} \mathcal{L}(n)^T y\right]^i \\
&= \left[-(\mathcal{M}_{\alpha\beta})^{\mu}_{\nu} n^{\nu} \mathcal{L}(n) \frac{\partial}{\partial n^{\mu}} \mathcal{L}(n)^T + \mathcal{L}(n) \mathcal{M}_{\alpha\beta} \mathcal{L}(n)^T\right]^{ij} y_j
\end{aligned} \tag{2.19}$$

where we have used the fact that

$$\mathcal{L}(n)\mathcal{L}(n)^T = 1 \quad \Longrightarrow \quad \left(\frac{\partial}{\partial n^{\mu}} \mathcal{L}(n)\right) \mathcal{L}(n)^T + \mathcal{L}(n) \frac{\partial}{\partial n^{\mu}} \mathcal{L}(n)^T = 0\,. \tag{2.20}$$

Thus, we find that

$$\sum_{i=5}^{8} \frac{\partial \xi^i}{\partial \omega^{\alpha\beta}}\bigg|_{\omega=0} \frac{\partial}{\partial \xi^i} = \left[\mathcal{L}(n) \mathcal{M}_{\alpha\beta} \mathcal{L}(n)^T - (\mathcal{M}_{\alpha\beta})^{\mu}{}_{\nu} n^{\nu} \mathcal{L}(n) \frac{\partial}{\partial n^{\mu}} \mathcal{L}(n)^T\right]^{\rho\sigma} y_{\sigma} \frac{\partial}{\partial y^{\rho}}\,. \tag{2.21}$$

Using equation (1.16) for $\mathcal{M}_{\alpha\beta}$, we obtain

$$\sum_{i=5}^{8} \frac{\partial \xi^i}{\partial \omega^{\alpha\beta}}\bigg|_{\omega=0} \frac{\partial}{\partial \xi^i} = \mathcal{L}_{\sigma\beta} \mathcal{L}^{\rho}{}_{\alpha} \left(y^{\sigma} \frac{\partial}{\partial y^{\rho}} - y^{\rho} \frac{\partial}{\partial y^{\sigma}}\right) - n_{\beta} \mathcal{L}^{\rho}{}_{\zeta} \frac{\partial}{\partial n^{\alpha}} \mathcal{L}_{\sigma}{}^{\zeta} \left(y^{\sigma} \frac{\partial}{\partial y^{\rho}} - y^{\rho} \frac{\partial}{\partial y^{\sigma}}\right) \tag{2.22}$$

which was called $g(\lambda_{\alpha\beta})$ in [6]. So finally, we obtain

$$X_{\alpha\beta} = \mathcal{L}_{\sigma\beta} \mathcal{L}^{\rho}{}_{\alpha} \left(y^{\sigma} \frac{\partial}{\partial y^{\rho}} - y^{\rho} \frac{\partial}{\partial y^{\sigma}}\right) - n_{\beta} \mathcal{L}^{\rho}{}_{\zeta} \frac{\partial}{\partial n^{\alpha}} \mathcal{L}_{\sigma}{}^{\zeta} \left(y^{\sigma} \frac{\partial}{\partial y^{\rho}} - y^{\rho} \frac{\partial}{\partial y^{\sigma}}\right) + n_{\beta} \frac{\partial}{\partial n^{\alpha}} - n_{\alpha} \frac{\partial}{\partial n^{\beta}} \tag{2.23}$$

which was called $ih_n(\lambda_{\alpha\beta})$ in [6]. It was shown that these generators satisfy the Lie algebra of $SL(2, C)$. We will maintain the matrix notation for $\mathcal{M}_{\alpha\beta}$ so that (2.23) may be written as

$$\begin{aligned}
X_{\alpha\beta} &= [\mathcal{L}(n) \mathcal{M}_{\alpha\beta} \mathcal{L}^T]^{\mu}{}_{\nu} y^{\nu} \frac{\partial}{\partial y^{\mu}} - \left[\mathcal{L}(\mathcal{M}_{\alpha\beta})^{\rho}{}_{\sigma} n^{\sigma} \frac{\partial}{\partial n^{\rho}} \mathcal{L}^T\right]^{\mu}{}_{\nu} y^{\nu} \frac{\partial}{\partial y^{\mu}} - (\mathcal{M}_{\alpha\beta})^{\rho}{}_{\sigma} n^{\sigma} \frac{\partial}{\partial n^{\rho}} \\
&= -y^T [\mathcal{L}(n) \mathcal{M}_{\alpha\beta} \mathcal{L}^T] \nabla_y - y^T \mathcal{L}(n) [n^T \mathcal{M}_{\alpha\beta} \nabla_n] \mathcal{L}^T \nabla_y - n^T \mathcal{M}_{\alpha\beta} \nabla_n
\end{aligned} \tag{2.24}$$

where $(\nabla_y)_{\mu} = \frac{\partial}{\partial y^{\mu}}$. By defining the 4-matrices

$$S_{\mu} = \mathcal{L} \frac{\partial}{\partial n^{\mu}} \mathcal{L}^T \qquad \mu = 0, \ldots, 3 \tag{2.25}$$

(which by (2.20) are antisymmetric) equation (2.24) becomes

$$X_{\alpha\beta} = -\left\{y^T[\mathcal{L}(n)\mathcal{M}_{\alpha\beta}\mathcal{L}^T]\nabla_y + n_\mu(\mathcal{M}_{\alpha\beta})^{\mu\nu}[y^T S_\nu \nabla_y + (\nabla_n)_\nu]\right\}. \qquad (2.26)$$

3. Classical and quantum mechanics of the generalized phase space

For classical dynamical systems whose potential depends only on ρ (given by (1.1)), we would like to write a Lagrangian for the reduced '1-body problem' which includes an explicit kinetic term for n. A possible choice is

$$\mathrm{L} = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}\lambda\dot{n}^2 - V(x^2) \qquad (3.1)$$

where λ is a length scale required because n is a unit vector. Notice that when $\dot{n} = 0$, the dynamics depend only on $\dot{x}$ for fixed n_μ and so the relative coordinate remains within RMS(n). Rewriting (2.9) as

$$\dot{x} = \mathcal{L}^T\dot{y} + \dot{\mathcal{L}}^T y = \mathcal{L}^T[\dot{y} + \mathcal{L}\dot{\mathcal{L}}^T y] \qquad (3.2)$$

we may write (3.1) in the form

$$\mathrm{L} = \tfrac{1}{2}m[\dot{y} + \mathcal{L}\dot{\mathcal{L}}^T y]^2 + \tfrac{1}{2}\lambda\dot{n}^2 - V(x^2). \qquad (3.3)$$

By construction, (3.3) is Lorentz invariant, and so is invariant under the transformations induced by (2.26). Therefore, applying Noether's theorem

$$\begin{aligned} 0 = \delta\mathrm{L} &= \frac{\partial\mathrm{L}}{\partial\xi^i}\delta\xi^i + \frac{\partial\mathrm{L}}{\partial\dot{\xi}^i}\delta\dot{\xi}^i \\ &= \frac{\partial\mathrm{L}}{\partial\xi^i}\delta\xi^i + \frac{\partial\mathrm{L}}{\partial\dot{\xi}^i}\frac{\mathrm{d}}{\mathrm{d}\tau}\delta\xi^i \\ &= \left[\frac{\partial\mathrm{L}}{\partial\xi^i} - \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\mathrm{L}}{\partial\dot{\xi}^i}\right]\delta\xi^i + \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{\partial\mathrm{L}}{\partial\dot{\xi}^i}\delta\xi^i\right] \end{aligned} \qquad (3.4)$$

where the first term vanishes for solutions to the Euler–Lagrange equation, and taking the variation to be $\delta\xi^i = \frac{1}{2}\omega^{\alpha\beta}X_{\alpha\beta}\,\xi^i$, one obtains the conservation law

$$\frac{\mathrm{d}}{\mathrm{d}\tau}[p^\mu X_{\alpha\beta}y_\mu + \pi^\mu X_{\alpha\beta}n_\mu] = 0 \qquad (3.5)$$

where

$$p_\mu = \frac{\partial\mathrm{L}}{\partial\dot{y}^\mu} \qquad \text{and} \qquad \pi_\mu = \frac{\partial\mathrm{L}}{\partial\dot{n}^\mu} \qquad (3.6)$$

using the notation p_μ for the variable conjugate to y^μ (for each n^μ). Since the variables y^μ are bounded by the RMS parametrization (1.7), the p_μ are symmetric but not self-adjoint. These operators, however, occur in combinations which have self-adjoint extensions. We discuss these questions elsewhere. Using equation (2.26) for $X_{\alpha\beta}$, (3.5) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\{y^T\mathcal{L}(n)\mathcal{M}_{\alpha\beta}\mathcal{L}^T p + n_\mu(\mathcal{M}_{\alpha\beta})^{\mu\nu}[y^T S_\nu p + \pi_\nu]\} = 0. \qquad (3.7)$$

If we understand π_ν, in the Poisson bracket sense, as a derivative with respect to n_μ, then the quantum operators $h_n(\lambda_{\alpha\beta})$ of [6] now appear as classical constants of the motion for the Lagrangian (3.1).

To obtain the Hamiltonian, we first observe that $\mathcal{L}$ depends on τ only through n, so

$$\mathcal{L}\dot{\mathcal{L}}^T = \mathcal{L}\left(\dot{n}^\nu\frac{\partial}{\partial n^\nu}\mathcal{L}^T\right) = \dot{n}^\nu S_\nu. \qquad (3.8)$$

Applying equation (3.6) to (3.3),

$$p_\mu = \frac{\partial L}{\partial \dot{y}^\mu} = m[\dot{y}_\mu + (\mathcal{L}\dot{\mathcal{L}}^T y)_\mu] \quad \Rightarrow \quad p = m[\dot{y} + \dot{n}^\nu S_\nu y] \tag{3.9}$$

and

$$\pi_\mu = \frac{\partial L}{\partial \dot{n}^\mu} = \lambda \dot{n}_\mu + m[\dot{y} + \dot{n}^\nu S_\nu y]^T \frac{\partial}{\partial \dot{n}^\mu}[\dot{y} + \dot{n}^\nu S_\nu y] = \lambda \dot{n}_\mu - y^T S_\mu p \tag{3.10}$$

where we used (3.9) and the antisymmetry of S_μ to obtain (3.10). Equations (3.9) and (3.10) may be inverted to eliminate $(\dot{n}, \dot{y})$:

$$\dot{n}_\mu = \frac{1}{\lambda}[\pi_\mu + y^T S_\mu p] \tag{3.11}$$

and

$$\dot{y} = \frac{1}{m} p - \dot{n}^\mu S_\mu y = \frac{1}{m} p - \frac{1}{\lambda}[\pi^\mu + y^T S^\mu p] S_\mu y \tag{3.12}$$

which may be used to write the Hamiltonian as

$$\begin{aligned} K &= \dot{y} \cdot p + \dot{n} \cdot \pi - L \\ &= p^T \left(\frac{1}{m} p - \frac{1}{\lambda}[\pi^\mu + y^T S^\mu p] S_\mu y \right) + \left(\frac{1}{\lambda}[\pi_\mu + y^T S_\mu p] \right) \pi^\mu - \tfrac{1}{2} m \left(\frac{1}{m^2} p^2 \right) \\ &\quad - \tfrac{1}{2}\lambda \left[\left(\frac{1}{\lambda} \right)^2 (\pi^\mu + y^T S^\mu p)(\pi_\mu + y^T S_\mu p) \right] + V \\ &= \frac{p^2}{2m} + \frac{1}{2\lambda}(\pi^\mu + y^T S^\mu p)(\pi_\mu + y^T S_\mu p) + V \, . \end{aligned} \tag{3.13}$$

Since S^μ is antisymmetric, we may regard (3.13) as a quantum Hamiltonian without ordering ambiguity in the operator $y^T S^\mu p$. The Schrödinger equation is then

$$\mathrm{i}\partial_\tau \psi = K \psi = \left[\frac{p^2}{2m} + \frac{1}{2\lambda}(\pi^\mu + y^T S^\mu p)(\pi_\mu + y^T S_\mu p) + V \right] \psi \tag{3.14}$$

where we take as quantum operators

$$p_\mu = \mathrm{i}\frac{\partial}{\partial y^\mu} \qquad \pi_\mu = \mathrm{i}\frac{\partial}{\partial n^\mu} \, . \tag{3.15}$$

We require that (3.14) be locally gauge invariant in the coordinate space (n, y), that is, under transformations of the form

$$\psi \longrightarrow \mathrm{e}^{-\mathrm{i}e\Theta(n,y)} \, \psi \tag{3.16}$$

this can be accomplished through the minimal coupling prescription

$$p_\mu \longrightarrow p_\mu - eA_\mu^{(n)} \qquad \pi_\mu \longrightarrow \pi_\mu - e\chi_\mu \tag{3.17}$$

together with the requirement that under gauge transformation

$$A_\mu^{(n)} \longrightarrow A_\mu^{(n)} + \frac{\partial}{\partial y^\mu}\Theta \qquad \chi_\mu \longrightarrow \chi_\mu + \left(\frac{\partial}{\partial n^\mu} + y^T S_\mu \nabla_y \right) \Theta \, . \tag{3.18}$$

Note that $A_\mu^{(n)}$ transforms under $O(3,1)$ as an induced (over $O(2,1)$) representation; it transforms as p_μ under Lorentz transformations (i.e. under the $O(2,1)$ little group) and so, since the Maxwell equations are Lorentz invariant, it satisfies the Maxwell equation in the y^μ variables. Under the gauge transformation

$$(p - eA^{(n)\prime})\mathrm{e}^{-\mathrm{i}e\Theta}\psi = \mathrm{e}^{-\mathrm{i}e\Theta}(p + e\nabla_y \Theta - eA^{(n)\prime})\psi = \mathrm{e}^{-\mathrm{i}e\Theta}(p - eA^{(n)})\psi \tag{3.19}$$

and

$$(\pi_\mu + y^T S_\mu p - e\chi'_\mu)e^{-ie\Theta}\psi = e^{-ie\Theta}\left(\pi_\mu + y^T S_\mu p + e\frac{\partial}{\partial n^\mu}\Theta + ey^T S_\mu \nabla_n \Theta - e\chi'_\mu\right)\psi$$
$$= e^{-ie\Theta}(\pi_\mu + y^T S_\mu p - e\chi_\mu)\psi \tag{3.20}$$

so that the gauge invariant form of (3.14) is

$$i\partial_\tau \psi = K\psi = \left[\frac{1}{2m}(p - eA^{(n)})^2 + \frac{1}{2\lambda}(\pi^\mu + y^T S^\mu p - e\chi^\mu)(\pi_\mu + y^T S_\mu p - e\chi_\mu) + V\right]\psi\,. \tag{3.21}$$

Consider the derivative operator which acts on $\Theta(n, y)$ in the transformation of the gauge field χ_μ in (3.18). We denote this operator by

$$D_\mu = (\nabla_n)_\mu + y^T S_\mu \nabla_y \tag{3.22}$$

and we notice that D_μ also appears in the Lorentz generators $X_{\alpha\beta}$ (2.26). From (3.11) we see that D_μ may be regarded as the quantum operator corresponding to $\lambda\dot{n}$. Using equation (3.22) in (2.26), the generators assume the simpler form

$$X_{\alpha\beta} = -\{y^T[\mathcal{L}(n)\mathcal{M}_{\alpha\beta}\mathcal{L}^T]\nabla_y + n_\mu(\mathcal{M}_{\alpha\beta})^{\mu\nu}D_\nu\}$$
$$= -\{x^T\mathcal{M}_{\alpha\beta}\nabla_x + n_\mu(\mathcal{M}_{\alpha\beta})^{\mu\nu}D_\nu\} \tag{3.23}$$

which, in light of (3.11) and the definitions of p_μ and π_μ, suggests the analogue

$$X_{\alpha\beta} \sim i[x^T\mathcal{M}_{\alpha\beta}(m\dot{x}) + n^T\mathcal{M}_{\alpha\beta}(\lambda\dot{n})]\,. \tag{3.24}$$

In fact, using (3.9) and (3.11) in (3.7), we find for the classical conservation law, that

$$\frac{d}{d\tau}\left\{y^T\mathcal{L}(n)\mathcal{M}_{\alpha\beta}\mathcal{L}^T p + n_\mu(\mathcal{M}_{\alpha\beta})^{\mu\nu}[y^T S_\nu p + \pi_\nu]\right\} = 0$$
$$= \frac{d}{d\tau}\left\{m\ y^T\mathcal{L}(n)\mathcal{M}_{\alpha\beta}\mathcal{L}^T[\dot{y} + \dot{n}^\nu S_\nu y] + n^T(\mathcal{M}_{\alpha\beta})[\lambda\dot{n}]\right\}$$
$$= \frac{d}{d\tau}\left\{m\ y^T\mathcal{L}(n)\mathcal{M}_{\alpha\beta}\mathcal{L}^T[\dot{y} + \mathcal{L}\dot{\mathcal{L}}^T y] + n^T(\mathcal{M}_{\alpha\beta})[\lambda\dot{n}]\right\}$$
$$= \frac{d}{d\tau}\left\{m\ x^T\mathcal{M}_{\alpha\beta}[\mathcal{L}^T\dot{y} + \dot{\mathcal{L}}^T y] + n^T(\mathcal{M}_{\alpha\beta})[\lambda\dot{n}]\right\}$$
$$= \frac{d}{d\tau}\left\{x^T\mathcal{M}_{\alpha\beta}[m\dot{x}] + n^T(\mathcal{M}_{\alpha\beta})[\lambda\dot{n}]\right\} \tag{3.25}$$

providing the generators with the form of a generalized angular momentum in terms of the relative Minkowski variables and the frame orientation variables.

The Hamiltonian (3.13) also assumes a simple form when expressed in terms of (3.22):

$$K = -\frac{1}{2m}\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial y_\mu} - \frac{1}{2\lambda}D_\mu D^\mu + V\,. \tag{3.26}$$

Suppose that a function $f(n, y)$ is defined in such a way that its dependence on n is only through $\mathcal{L}(n)^T y$ (which is to say that f is a function of x alone, even as n varies in τ). Then we find that

$$\frac{\partial}{\partial y^\mu}f = \left.\frac{df}{d\xi^\alpha}\right|_{\xi=\mathcal{L}(n)^T y}\frac{\partial}{\partial y^\mu}(\mathcal{L}_\beta{}^\alpha y^\beta) = \mathcal{L}_\mu{}^\alpha\left.\frac{df}{d\xi^\alpha}\right|_{\xi=\mathcal{L}(n)^T y} \tag{3.27}$$

and

$$\frac{\partial}{\partial n^\mu}f = \left.\frac{df}{d\xi^\alpha}\right|_{\xi=\mathcal{L}(n)^T y}\frac{\partial}{\partial n^\mu}(\mathcal{L}_\beta{}^\alpha y^\beta) \tag{3.28}$$

so that

$$
\begin{aligned}
D_\mu f &= \left(\frac{\partial}{\partial n^\mu} + y^T S_\mu \nabla_y\right) f \\
&= \left[\frac{\partial}{\partial n^\mu} + y_\beta \mathcal{L}^\beta{}_\gamma \left(\frac{\partial}{\partial n^\mu}\mathcal{L}^{\alpha\gamma}\right)\frac{\partial}{\partial y^\alpha}\right] f \\
&= \left.\frac{\mathrm{d}f}{\mathrm{d}\xi^\sigma}\right|_{\xi=\mathcal{L}(n)^T y} y^\beta \left[\frac{\partial}{\partial n^\mu}\mathcal{L}_\beta{}^\sigma + \mathcal{L}_\beta{}^\gamma (\frac{\partial}{\partial n^\mu}\mathcal{L}^\alpha{}_\gamma)\mathcal{L}_\alpha{}^\sigma\right] \\
&= \left.\frac{\mathrm{d}f}{\mathrm{d}\xi^\sigma}\right|_{\xi=\mathcal{L}(n)^T y} y^\beta \left[\frac{\partial}{\partial n^\mu}\mathcal{L}_\beta{}^\sigma + \mathcal{L}_\beta{}^\gamma (\mathcal{L}^T)^\sigma{}_\alpha \frac{\partial}{\partial n^\mu}\mathcal{L}^\alpha{}_\gamma\right] \\
&= \left.\frac{\mathrm{d}f}{\mathrm{d}\xi^\sigma}\right|_{\xi=\mathcal{L}(n)^T y} y^\beta \left[\frac{\partial}{\partial n^\mu}\mathcal{L}_\beta{}^\sigma - \mathcal{L}_\beta{}^\gamma \mathcal{L}^\alpha{}_\gamma \frac{\partial}{\partial n^\mu}\mathcal{L}_\alpha{}^\sigma\right] \\
&\equiv 0
\end{aligned}
\tag{3.29}
$$

where we have used (2.20). Thus, D_μ acts as a kind of covariant derivative which vanishes on functions of x alone. In particular, D_μ vanishes on the eigenstates discussed in [5, 6], in which case the Hamiltonian (3.13) and (3.26) reduces to the RMS Hamiltonian discussed in [5]. The dynamical effects that we shall discuss in the next section are associated with the evolution of the wavefunction of the system to a form which does not depend only on x^μ.

Notice also that

$$
\begin{aligned}
\dot n^\mu D_\mu &= \left(\dot n^\mu \frac{\partial}{\partial n^\mu} + y^T \dot n^\mu S_\mu \nabla_y\right) \\
&= \left(\dot n \cdot \nabla_n - y^T \dot{\mathcal{L}}\mathcal{L}^T \nabla_y\right) \\
&= \left(\dot n \cdot \nabla_n - \left[\frac{d}{d\tau}(\mathcal{L}^T y) - \dot y^T \mathcal{L}\right]\mathcal{L}^T \nabla_y\right) \\
&= \left(\dot n \cdot \nabla_n + \dot y \cdot \nabla_y - \dot x \cdot \nabla_x\right) .
\end{aligned}
\tag{3.30}
$$

We may rewrite this expression as

$$
\mathrm{d}x \cdot \nabla_x + \mathrm{d}n^\mu D_\mu = \mathrm{d}y \cdot \nabla_y + \mathrm{d}n \cdot \nabla_n \tag{3.31}
$$

which shows in yet another way that ∇_x and D_μ generate the changes induced by $\mathrm{d}x$ and $\mathrm{d}n$ (with x^μ held constant), just as ∇_y and ∇_n generate the changes induced by $\mathrm{d}y$ and $\mathrm{d}n$ (with y^μ held constant).

It will be useful to examine the classical Lagrangian in the presence of the fields $A_\mu^{(n)}$ and χ_μ, which we may find by treating the Hamiltonian in (3.21) as a classical functional and evaluating

$$
\dot n^\mu = \frac{\partial}{\partial \pi_\mu} K = \frac{1}{\lambda}(\pi^\mu + y^T S^\mu p - e\chi^\mu) \tag{3.32}
$$

and

$$
\begin{aligned}
\dot y_\mu &= \frac{\partial}{\partial p^\mu} K = \frac{1}{m}(p_\mu - eA_\mu^{(n)}) + \frac{1}{\lambda}(\pi_\nu + y^T S_\nu p - e\chi_\nu)\frac{\partial}{\partial \pi^\mu}(y^T S^\nu p) \\
&= \frac{1}{m}(p_\mu - eA_\mu^{(n)}) - \dot n_\nu (S^\nu)_{\mu\sigma} y^\sigma .
\end{aligned}
\tag{3.33}
$$

Recalling (3.8), we find that

$$
\begin{aligned}
L &= p \cdot \dot y + \pi \cdot \dot n - K \\
&= \tfrac{1}{2} m[\dot y + \mathcal{L}\dot{\mathcal{L}}^T y]^2 + \tfrac{1}{2}\lambda \dot n^2 + e[(\dot y + \mathcal{L}\dot{\mathcal{L}}^T y) \cdot A^{(n)} + \dot n \cdot \chi] - V(x^2) .
\end{aligned}
\tag{3.34}
$$

From equation (3.2), we have

$$\dot{y} + \mathcal{L}\dot{\mathcal{L}}^T y = \mathcal{L}\dot{x} \tag{3.35}$$

so that we may write (3.34) in the form

$$L = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}\lambda\dot{n}^2 + e[\dot{x}\cdot(\mathcal{L}^T A^{(n)}) + \dot{n}\cdot\chi] - V(x^2)\,. \tag{3.36}$$

In order for L to be a Lorentz scalar, $\mathcal{L}^T A^{(n)}$ must transform under the full Lorentz group $O(3,1)$. Since $A^{(n)}$ was introduced as a field which transforms under the $O(2,1)$ little group, we may write

$$A^{(n)\prime} = D^{-1}(\Lambda, n)A^{(n)} = \mathcal{L}(\Lambda n)\,\Lambda\,\mathcal{L}^T(n)A^{(n)} \Longrightarrow \mathcal{L}^T(\Lambda n)A^{(n)\prime} = \Lambda\,\mathcal{L}^T(n)A^{(n)} \tag{3.37}$$

verifying that the combination $\mathcal{L}^T A^{(n)}$ transforms as a 4-vector under Λ.

4. The Zeeman effect

In [6], the spacelike vector n played no particular role in the dynamics and could be chosen arbitrarily, because the systems under discussion were $O(3,1)$-symmetric and no direction in spacetime was intrinsic to the problem (other than the axis of the bound state). That situation generalizes the non-relativistic spherically symmetric central force problem, in which the absence of a preferred direction in space leads to the degeneracy of the energy spectrum with respect to the magnetic quantum number (which characterizes the orientation of the angular momentum). In [7], it was shown that the vector n plays a role in dipole radiation from the bound state, because conservation of angular momentum and the spin-1 nature of the electromagnetic field impose an orientation dependence on the interaction. Thus, the photon carries off spin provided by the bound-state transition, and that transition depends on the orientation of the angular momentum of the state (determined by n) and the photon polarization.

In the Zeeman effect, one lifts the degeneracy of the bound-state spectrum by placing the state in a constant external magnetic field, which interacts with the magnetic moment (angular momentum) of the system and thereby provides a preferred direction in space. In the semiclassical picture, the atom will tend to rotate. The interaction angular momentum is intimately connected with the rotation generators, and for the bound states discussed here, these generators are elements of the rotation subgroup of the induced representation of $O(3,1)$. Since the rotation group $O(3) \subset O(3,1)$ acts on the vector n as well as the RMS variables y, the relativistic Zeeman effect can clearly only be described in the context of a theory which explicitly permits the generators to act directly on all the variables in the theory. In this section, we provide such a description in the context of the Hamiltonian theory given in the section 4.

In the non-relativistic case, the Zeeman effect is obtained as a first order perturbation of the hydrogen atom bound state, by a vector potential

$$\boldsymbol{A}(\boldsymbol{r}) = -\tfrac{1}{2}\boldsymbol{B}\times\boldsymbol{r} \tag{4.1}$$

which leads to the constant magnetic field

$$(\nabla\times\boldsymbol{A})^i = \epsilon^{ijk}\frac{\partial}{\partial r^j}\left(-\tfrac{1}{2}\epsilon_{klm}B_l r_m\right) = B^i\,. \tag{4.2}$$

The Hamiltonian becomes

$$\begin{aligned} H &= \frac{1}{2m}(p - eA)^2 + V \\ &= \frac{p^2}{2m} + V + \frac{e}{2m}(p \cdot A + A \cdot p) + o(e^2) \\ &= H_0 + \frac{e}{m}A \cdot p + o(e^2) \\ &= H_0 - \frac{e}{2m}(B \times r) \cdot p + o(e^2) \\ &= H_0 - \frac{e}{2m}B \cdot (r \times p) + o(e^2) \\ &= H_0 - \frac{e}{2m}B \cdot L + o(e^2) \end{aligned} \tag{4.3}$$

where $L = r \times p$ is the angular momentum operator. Thus taking B in the direction of the diagonal angular momentum operator (usually the z-axis), the observed Zeeman splitting is obtained from (4.3) as

$$E_{ln} \longrightarrow E_{lnq} = E_{ln} - \frac{eB}{2m}q\,. \tag{4.4}$$

where q is the eigenvalue of the operator L_z.

In section 3, we introduced two gauge compensation fields, $A_\mu^{(n)}$ and χ_μ, required to make the Hamiltonian (3.13) locally gauge invariant. However, we now argue that just as n and y transform under inequivalent representations of the Lorentz group (y transforms under the $O(2,1)$ little group induced by the action of the full $O(3,1)$), so $A_\mu^{(n)}$ and χ_μ must be seen as inequivalent representations of the usual $U(1)$ gauge group of electromagnetism. In the full spacelike region, a constant electromagnetic field, $F^{\mu\nu}$, can be represented through the vector potential

$$A^\mu(x) = -\tfrac{1}{2}F^{\mu\nu}x_\nu\,. \tag{4.5}$$

We now restrict the support of A^μ to $x \in \mathrm{RMS}(n)$ and express the vector potential as a vector oriented with $\mathrm{RMS}(\hat{n})$ by writing

$$A_\mu^{(n)}(y) = \mathcal{L}_{\mu\nu}A^\nu(\mathcal{L}^T y) = -\tfrac{1}{2}\mathcal{L}_{\mu\nu}F^\nu{}_\sigma\mathcal{L}_\lambda{}^\sigma y^\lambda = -\tfrac{1}{2}(\mathcal{L}F\mathcal{L}^T y)_\mu\,. \tag{4.6}$$

For the field χ_μ, we choose (note that n undergoes Lorentz transform in the same way as x),

$$\chi_\mu(n) = b^2 A_\mu(n) = -\tfrac{1}{2}b^2 F^\nu{}_\sigma n^\sigma \tag{4.7}$$

(here b is another length scale, required since $A_\mu(x)$ has units of length^{-1}, so F_σ^ν must have units of length^{-2}, but χ_μ must be without units) and we use (4.6) and (4.7) in the Schrödinger equation (3.21):

$$\begin{aligned} \mathrm{i}\partial_\tau\psi &= \left[\frac{1}{2m}(p - eA^{(n)})^2 + \frac{1}{2\lambda}(\pi^\mu + y^T S^\mu p - e\chi^\mu)(\pi_\mu + y^T S_\mu p - e\chi_\mu) + V\right]\psi \\ &= \left[\frac{1}{2m}p^2 - \frac{e}{2m}(p \cdot A^{(n)} + A^{(n)} \cdot p) + \frac{1}{2\lambda}(\pi^\mu + y^T S^\mu p)^2\right. \\ &\quad \left. - \frac{e}{2\lambda}[(\pi^\mu + y^T S^\mu p)\chi_\mu + \chi^\mu(\pi_\mu + y^T S_\mu p)] + V + o(e^2)\right]\psi \\ &= \left[\frac{1}{2m}p^2 + \frac{1}{2\lambda}(\pi^\mu + y^T S^\mu p)^2 + V\right. \\ &\quad \left. - e\left[\frac{1}{m}A^{(n)} \cdot p + \frac{1}{\lambda}\chi^\mu(\pi_\mu + y^T S_\mu p)\right] + o(e^2)\right]\psi \end{aligned} \tag{4.8}$$

where the first three terms of (4.8) are the unperturbed Hamiltonian K_0.

The perturbation term to order o(e), is

$$-e\left[\frac{1}{m}A^{(n)}\cdot p+\frac{1}{\lambda}\chi^{\mu}(\pi_\mu+y^T S_\mu p)\right]$$
$$=-e\left[\frac{1}{m}A^{(n)T}p+\frac{1}{\lambda}\left[\chi^T\pi+y^T(S\cdot\chi)p\right]\right]$$
$$=-\frac{e}{2}\left[\frac{1}{m}(\mathcal{L}F\mathcal{L}^T y)^T p+\frac{b^2}{\lambda}F^{\mu}{}_{\nu}n^{\nu}(\pi_\mu+y^T S_\mu p)\right]$$
$$=\frac{e}{2m}\left[y^T\mathcal{L}F\mathcal{L}^T p+\frac{mb^2}{\lambda}n_\nu F^{\nu\mu}(\pi_\mu+y^T S_\mu p)\right]. \tag{4.9}$$

We now expand the electromagnetic field tensor on the basis of four by four antisymmetric tensors given by the Lorentz generators $\mathcal{M}^{\mu\nu}$. Thus

$$F=\tfrac{1}{2}F_{\mu\nu}\mathcal{M}^{\mu\nu} \tag{4.10}$$

may be verified through

$$(F)^{\alpha\beta}=\tfrac{1}{2}F_{\mu\nu}(\mathcal{M}^{\mu\nu})^{\alpha\beta}=\tfrac{1}{2}F_{\mu\nu}(g^{\mu\alpha}g^{\nu\beta}-g^{\mu\beta}g^{\nu\alpha})=F^{\alpha\beta}. \tag{4.11}$$

Using equation (4.10) in (4.9) we find that the perturbation term to order o(e) becomes

$$\frac{e}{4m}F_{\alpha\beta}\left[y^T\mathcal{L}\mathcal{M}^{\alpha\beta}\mathcal{L}^T p+\frac{mb^2}{\lambda}n_\mu(\mathcal{M}^{\alpha\beta})^{\mu\nu}(\pi_\nu+y^T S_\nu p)\right]. \tag{4.12}$$

We note that if $\lambda/b^2=m$, then we may write the first-order perturbation (using (3.23)) as

$$\frac{e}{4m}F_{\alpha\beta}[y^T\mathcal{L}\mathcal{M}^{\alpha\beta}\mathcal{L}^T p+n_\mu(\mathcal{M}^{\alpha\beta})^{\mu\nu}(\pi_\nu+y^T S_\nu p)]=\frac{e}{4m}F_{\alpha\beta}X^{\alpha\beta}. \tag{4.13}$$

For $F^{\mu\nu}F_{\mu\nu}=2(B^2-E^2)>0$, there exists a frame for which the interaction is purely magnetic. In such a frame, the perturbation becomes

$$\frac{e}{4m}F_{\alpha\beta}X^{\alpha\beta}=\frac{e}{4m}F_{ij}X^{ij}=\frac{e}{4m}\epsilon_{ijk}B^kX^{ij}=\frac{e}{2m}B^k\left[\tfrac{1}{2}\epsilon_{ijk}X^{ij}\right]=\frac{e}{2m}B^kh(\lambda_k) \tag{4.14}$$

where $h(\lambda_k)$ are the three conserved generators of the $SU(2)$ rotation subgroup of $SL(2,C)$ for the phase space $\{(n,y);(\pi,p)\}$, that is, the angular momentum operator for the eigenstates of the induced representation. Notice that in the matrix element for unperturbed eigenstates, the second terms of (4.9) vanishes, so the relativistic Zeeman effect does not depend upon the values of λ or b.

In [6], the diagonal angular momentum operator is $L_1(n)=h(\lambda_1)=-\mathrm{i}\partial/\partial\gamma$, and so if we take $B=B(1,0,0)$ then we find that

$$K_0\longrightarrow K=K_0-\frac{eB}{2m}h(\lambda_1) \tag{4.15}$$

splits the mass levels of the bound states according to

$$K'_{\ell n}\longrightarrow K'_{\ell n}-\frac{eB}{2m}q\,. \tag{4.16}$$

In going from (4.15) to (4.16), we have used the fact that the unperturbed Hamiltonian of (4.8) reduces to the unperturbed Hamiltonian of [6]. Equation (4.16) further justifies the conclusion reached in [7] that q is the magnetic quantum number. Moreover, the manifest covariance of the formalism guarantees that the splitting of the spectrum will be independent of the observer. We observe that if $F^{\mu\nu}F_{\mu\nu}<0$, we may find a frame in which the interaction is purely electric, leading to a covariant formulation of the Stark effect. Since the electric field couples to the boost generators (which reduce to the position operator in the non-relativistic limit) and these generators are not diagonal in this representation, the Stark effect remains formally (one really has only a resonance spectrum; the bound states are destroyed by the non-compact generator) a second-order perturbation, and we will discuss it elsewhere.

5. Multiplicity

In this paper we have studied the normal Zeeman effect for the case of two particles without spin. As pointed out in [6], the quantum number q belongs to a representation in the double covering of the Lorentz group, which takes on, in fact, a half-integer value, and indicates even multiplicity for the normal Zeeman splittings. One would expect, from the non-relativistic theory, to find odd multiplicity for the normal Zeeman effect. This result constitutes a prediction of the theory, concerned with the structure and physical relevance of the Lorentz group representation obtained, as in [6], by extracting the induced representation of $SL(2, C)$ on the $SU(1, 1)$ little group of a spacelike vector (the covering of $O(2, 1)$).

It is difficult to check this prediction at the present time, since two-body systems accessible to Zeeman splitting experiments generally consist of spin-$\frac{1}{2}$ particles. The non-relativistic limit of the Zeeman multiplicity is very delicate, since the relativistic representations are highly degenerate.

In the theory of spin-$\frac{1}{2}$ particles, the representation of the Lorentz group is constructed by induction from the little group of a timelike vector [9]. This construction can be generalized to two or more particles, using the same timelike vector for every particle (not the momentum, as in the Wigner construction [10]). The geometry of the case of two particles with spin is therefore quite different. The representation must be induced on the stability group of both a timelike and a spacelike vector, i.e. $U(1)$, and the differential equation imposing definite values for the Casimir operators, involving both vectors, have a very different structure. The multiple connectedness of the $O(2, 1)$ invariant spacetime support manifold of the two-body bound state no longer reflects the topology of the system.

This problem is currently under investigation.

6. Discussion

As in the non-relativistic case, the discussion of the Zeeman effect exposes the relationship of physics and geometry in the relativistic bound state. In [5], bound-state solutions were found for a Poincaré covariant generalization of the Schrödinger equation, with a mass spectrum corresponding to the non-relativistic energy spectrum (a notable failure of the Klein–Gordon theory with Ze^2/r potential [11]). A central feature of this model is that bound states are found only when the relative motion is restricted to the $O(2, 1)$ invariant region described by choosing an arbitrary spacelike direction n_μ and requiring $(x_\perp)^2 = [x - (x \cdot n)n]^2 \geqslant 0$. Mathematically, this restriction is related to the existence of discrete unitary representations of the Lorentz group in this subspace [12]. Physically, this restriction leads to a lowering of the mass spectrum (cf [13]). Since these states form a representation of $SU(1, 1)$, it is possible to construct an induced representation of the full Lorentz group [6] by studying the action of Lorentz transformations on x and n together. Thus, the $O(2, 1)$ stabilized direction n was seen to have kinematic—but not dynamical—significance, and remained arbitrary. However, in the study of dipole radiation from these bound states [7], it was shown that the diagonal component of angular momentum, related to n_μ, would shift as a result of photon emission. We argued that the component of spin carried off by the photon is provided by the bound state, through the action of the $O(3)$ rotation subgroup of the induced $O(3, 1)$, and therefore the photon must couple to n_μ.

In this paper, we have examined the role of the spacelike direction n_μ, by constructing a model in which n_μ is treated as a dynamical quantity at the classical and quantum levels. We build a configuration space consisting of the subspace orientation n_μ and the parametrization variables of the subspace y^μ, and show that the generators of the Lorentz group on this space are identical to those obtained for the induced representation of $O(3, 1)$ in [6]. We write a model Lagrangian containing a kinetic term for n_μ and Legendre transform it to the Hamiltonian, whose quantum form is seen to contain the Hamiltonian used in [5], plus a term quadratic in a covariant derivative which vanishes on the bound states of [5, 6]. By making this Hamiltonian locally gauge invariant, we introduce an interaction with the gauge field which couples to the momenta conjugate to y^μ and n_μ, thereby giving the orientation n_μ a dynamical role. When the gauge potential is taken to represent a constant electromagnetic field, we find that the field interacts with the generators of the Lorentz group in much the same manner that the magnetic field interacts with the rotation generators in the corresponding non-relativistic problem. From these developments, it is clear that the only way to couple an external magnetic field to the magnetic moment of the bound state (represented by the rotation subgroup of the induced representation of the Lorentz group) is by treating n_μ as a dynamical quantity. In the splitting, obtained here in a covariant form independent of the observer's frame, the external magnetic field is coupled to q, the observable component of the angular momentum, which is given entirely by n_μ and its derivatives. In the non-relativistic Zeeman effect, the degeneracy of the energy levels associated with the rotation invariance of the bound state is lifted by placing the state in a magnetic field which provides a preferred direction in space. In the relativistic treatment presented here, the degeneracy of the mass levels associated with the Lorentz invariance of the bound state is lifted by placing the state in a magnetic field which provides a preferred direction in spacetime.

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